Solutions of semestral examination, probability theory I, B. math. I year, 2008-09, ISI-BC

Solution 1:
(a) The sample space will be, the set of infinite sequences which have the elements from the set $\{1, 2, \cdots, n\}$.

(b)

$$q_r \equiv P(R = r) = \frac{\binom{n}{r-1}(r-1)!\binom{r-1}{1}}{n^r} = \frac{r-1}{n^r}\frac{n!}{(n-r+1)!}.$$

(c)

$$P_r = P(R > r) = \frac{\binom{n}{r}r!}{n^r} = \frac{(n)_r}{n^r}. \tag{1}$$

$$\begin{aligned} P_1 &= P(R > 1) \\ &= q_2 + q_3 + \cdots + q_{n+1} \\ &= \frac{(n)_1}{n^1} \qquad \text{[from (1)]} \\ &= 1. \end{aligned}$$

Solution 2:
(a) Let $n = p+q$, where there are p no. of $+1$s and q no. of -1s. So, $\omega_n = p-q = x$.

A path $(\omega_1, \omega_2, \cdots, \omega_n)$ from the origin to the point (n, x) is a polynomial line whose vertices have abscissas $0, 1, \cdots, n$ and ordinates $\omega_0, \omega_1, \cdots, \omega_n$ satisfying the given conditions for ω_i.

We consider n as the length of the path. There are 2^n paths of length n.

A path from the origin to an arbitrary point (n, x) exists only if n and x are of the form $n = p + q$ and $x = p - q$.

So p, $+1$s can be choosen from $n = p + q$ available places in

$$N_{n,x} = \binom{p+q}{p} = \binom{p+q}{q}$$

different ways. So, there exist exactly $N_{n,x}$ different paths from the origin to an arbitrary point (n, x).

(b) So, the event $E_{2n} = \{\omega : \omega_{2n} = 0\}$ i.e. all the paths from $(0,0)$ to $(2n,0)$ means return to the origin. From the previous argument of (a),

$$P(E_{2n}) = P(N_{2n,0}) = \binom{2n}{n} 2^{-2n}.$$

From Stirling's formula,

$$\binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}},$$

as $n \to \infty$. Hence, $\sqrt{\pi n} P(E_{2n}) \to 1$ as $n \to \infty$.

Solution 3:

(1) In an infinitesimal time interval dt there may occur only one event. This happens with the probability λdt independent of events outside the interval.

(2) The number of events $N(t)$ in a finite interval of length t obeys the $Poisson(\lambda t)$ distribution,

$$P\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Our aim is to show that $1 \Rightarrow 2$.

Assume that the events in different intervals are independent and

$$P(N_{t+dt} - N_t = 1) = \lambda dt \text{ and } P(N_{t+dt} = N_t) = 1 - \lambda dt.$$

Consider the generating function $\mathcal{G}_t(x)$:

$$\mathcal{G}_t(x) = E[x^{N(0,t)}]$$

$$\begin{aligned}
\mathcal{G}_{t+dt}(x) = E[x^{N(0,t+dt)}] &= E[x^{N(0,t)+N(t,t+dt)}] \\
&= E[x^{N(0,t)}]E[x^{N(t,t+dt)}] \\
&= \{\mathcal{G}_t(x)\}\{(1-\lambda dt)x^0 + \lambda dt x^1\} \\
&= \mathcal{G}_t(x) - \lambda dt(1-x)\mathcal{G}_t(x)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{\mathcal{G}_{t+dt}(x) - \mathcal{G}_t(x)}{dt} = \lambda(x-1)\mathcal{G}_t(x) \\
&\Rightarrow \frac{d}{dt}\mathcal{G}_t(x) = \lambda(x-1)\mathcal{G}_t(x) \\
&\Rightarrow \frac{d}{dt}\log \mathcal{G}_t(x) = \lambda(x-1) \\
&\Rightarrow \log \mathcal{G}_t(x) - \log \mathcal{G}_0(x) = \lambda(x-1)t \\
&\Rightarrow \log \mathcal{G}_t(x) - 0 = \lambda(x-1)t \\
&\Rightarrow \mathcal{G}_t(x) = e^{(x-1)\lambda t}
\end{aligned}$$

therefore the generating function is of the Poisson distribution.

Solution 4:

$X \sim N(0,1)$. The probability density function ϕ is given by $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, for

$x \in \mathbb{R}$.
(a) Note that, $\phi'(x) = -x\phi(x)$ and $\phi(x) \to 0$ as $x \to +\infty$ and $x \to -\infty$.

$$\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2\phi(x)dx \\
&= \int_{-\infty}^{\infty} x.x\phi(x)dx \\
&= -\int_{-\infty}^{\infty} x\phi'(x)dx \\
&= -[x\phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(x)dx \qquad \text{[integration by parts]} \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

Let it holds for $k = n$, therefore

$$E[X^{2n}] = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Now, for $k = n + 1$

$$\begin{aligned}
E[X^{2n+2}] &= \int_{-\infty}^{\infty} x^{2n+2}\phi(x)dx \\
&= \int_{-\infty}^{\infty} x^{2n+1}x\phi(x)dx \\
&= -\int_{-\infty}^{\infty} x^{2n+1}\phi'(x)dx \\
&= -[x^{2n+1}\phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n+1)x^{2n}\phi(x)dx \qquad \text{[integration by parts]} \\
&= 0 + (2n+1)E[X^{2n}] \\
&= 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1).
\end{aligned}$$

Hence the result.

(b)

$$\begin{aligned}
E[e^{sX}] &= \int_{-\infty}^{\infty} e^{sx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{sx-\frac{x^2}{2}}dx \\
&= e^{\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}+sx-\frac{x^2}{2}}dx \\
&= e^{\frac{s^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-s)^2}{2}}dx \\
&= e^{\frac{s^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \qquad \text{[by changing, } x-s=z\text{]} \\
&= e^{\frac{s^2}{2}}.
\end{aligned}$$

Solution 5:
(a)

$$\begin{aligned}\int_0^\infty P(X > x)dx &= \int_0^\infty \int_x^\infty f_X(t)dtdx \\ &= \int_0^\infty \int_0^t f_X(t)dxdt \\ &= \int_0^\infty tf_X(t)dt \\ &= E[X].\end{aligned}$$

(b) From (a)

$$E[X^n] = \int_0^\infty P(X^n > t)dt.$$

Now, by putting $t = x^n$,

$$\begin{aligned}E[X^n] &= \int_0^\infty nx^{n-1}P(X^n > x^n)dx \\ &= \int_0^\infty nx^{n-1}P(X > x)dx.\end{aligned}$$

Solution 6:
(a)

$$\begin{aligned}\int_{-\infty}^\infty f(x)dx &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{dx}{1+x^2} \\ &= \frac{1}{\pi}\big[\arctan x\big]_{-\infty}^\infty \\ &= \frac{1}{\pi}\big[\frac{\pi}{2} - (-\frac{\pi}{2})\big] \\ &= 1.\end{aligned}$$

And for all $x \in \mathbb{R}$,

$$f(-x) = f(x) = \frac{1}{\pi(1+x^2)}.$$

So,

$$P(X \geq 0) = \int_{-\infty}^0 \frac{dx}{\pi(1+x^2)} = \frac{1}{2} = \int_0^\infty \frac{dx}{\pi(1+x^2)} = P(X \leq 0).$$

Hence, the median is at 0.

(b)

$$\begin{aligned}P(Y \leq y) &= P(\frac{1}{X} \leq y) \\ &= P(X - X^2Y \leq 0) \\ &= P(X(1 - Xy) \leq 0).\end{aligned}$$

Therefore,

$$
\begin{aligned}
P(Y \leq y) &= P(\frac{1}{y} \leq X \leq 0); && \text{if } y > 0, \\
P(Y \leq y) &= \frac{1}{2}; && \text{if } y = 0, \\
P(Y \leq y) &= 1 - P(0 \leq X \leq \frac{1}{y}); && \text{if } y < 0.
\end{aligned}
$$

Now, for $y > 0$,

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}(F_X(0) - F_X(1/y)) \\
&= \frac{1}{y^2} f_X(1/y) \\
&= \frac{1}{\pi(1+y^2)}.
\end{aligned}
$$

Similarly we can show that for $y > 0$, $f_Y(y) = \frac{1}{\pi(1+y^2)}$.

For $y = 0$,

$$
\begin{aligned}
f_Y(0) &= \lim_{\varepsilon \to 0} \frac{F_Y(\varepsilon) - F_Y(-\varepsilon)}{2\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{2\int_0^\varepsilon \frac{dt}{\pi(1+t^2)}}{2\varepsilon} \\
&= \frac{1}{\pi(1+\varepsilon^2)}\Big|_{\varepsilon=0} \\
&= \frac{1}{\pi}.
\end{aligned}
$$

Hence the result.